

# A comparison theorem between Radon and Fourier-Laplace transforms for $\mathcal{D}$ -modules

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## Abstract

We prove a comparison theorem between the  $d$ -plane Radon transform and the Fourier-Laplace transform for  $\mathcal{D}$ -modules. This generalizes results of Brylinski and d'Agnolo-Eastwood.

## Introduction

The history of the Radon transform goes back to the famous paper [Rad17] of Radon. This transformation associates to a function  $f$  on  $\mathbb{R}^n$  a corresponding function on the family of affine  $d$ -dimensional planes in  $\mathbb{R}^n$  whose value at a given plane is the integral over  $f$  restricted to this plane. Since then various generalizations, like the Radon transform on homogeneous spaces [Hel66] and the Penrose transform [EPW81] were made, which had plenty of applications in representation theory, harmonic analysis and mathematical physics. Later it was realized that the Radon transform and its various cousins could be best understood in the context of integral geometry and  $\mathcal{D}$ -modules (see e.g. [D'A00]). This Radon transform for  $\mathcal{D}$ -modules was introduced by Brylinski in [Bry86]. There, he considers  $\mathcal{D}$ -modules on the complex projective space and measures their restriction to all  $d$ -planes, which gives rise to a (complex of)  $\mathcal{D}$ -module(s) on the corresponding Grassmanian. As an application he proved, among other things, the irreducibility of the monodromy action on the vanishing cohomology of a hyperplane section of a (possibly singular) variety.

The Fourier-Laplace transform on the other hand is an indispensable tool in the theory of differential equations. In the context of integral geometry it is a transform with a so-called exponential kernel. This property is reflected in the  $\mathcal{D}$ -module picture by the fact that the Fourier-Laplace transform does not preserve regular holonomicity. If however the  $\mathcal{D}$ -module is monodromic then

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the Fourier-Laplace transform is equivalent to the so-called Fourier-Sato transform (or monodromic Fourier-Laplace transform) which preserves regularity. By a theorem of Brylinski [Bry86], the hyperplane Radon transform for  $\mathcal{D}$ -modules on  $\mathbb{P}^n$  is closely related to the monodromic Fourier-Laplace transform on  $\mathbb{C}^{n+1}$ . Roughly speaking the theorem of Brylinski says, that the hyperplane Radon transform of a holonomic  $\mathcal{D}$ -module on  $\mathbb{P}^n$  is isomorphic to the (monodromic) Fourier-Laplace transform of a specific lift of the  $\mathcal{D}$ -module from  $\mathbb{P}^n$  to  $\mathbb{C}^{n+1}$ . This result was generalized by d'Agnolo and Eastwood [DE03] to quasi-coherent  $\mathcal{D}$ -modules and to a variant of the Radon transform which does measure the restriction of the  $\mathcal{D}$ -module to the complement of a given hyperplane, rather than the restriction to the hyperplane itself.

In this paper we extend the results of Brylinski and d'Agnolo-Eastwood to the case of the  $d$ -plane Radon transform. Let us give a short overview of the paper. We first give a brief review of algebraic  $\mathcal{D}$ -modules, the Radon as well as the Fourier-Laplace transform and state the comparison result for the hyperplane case of d'Agnolo and Eastwood. The proof of the general case proceeds as follows. We first prove that the  $d$ -plane Radon transform for a  $\mathcal{D}$ -module is equivalent to a diagonal embedding of this  $\mathcal{D}$ -module on a product of projective spaces and then applying the hyperplane Radon transform on each factor (Lemma 2.6 and Lemma 2.8). This result stems from the simple geometric fact that a  $d$ -plane in  $\mathbb{P}^n$  is isomorphic to the diagonal in  $(\mathbb{P}^n)^{\times n-d}$  cut with an appropriate hyperplane on each factor. We then prove in Proposition 2.9 that we can apply the theorem of d'Agnolo and Eastwood on each factor. Finally we have to prove that the extension functors and the Fourier-Laplace transform interchange (Lemma 2.10).

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## 1 Preliminaries

In the first section we review briefly the theory of algebraic  $\mathcal{D}$ -modules in order to fix notations. In the second section we review the  $d$ -plane Radon transform and various variants (cf. Definition 1.1) and show how they arise as an integral transformation (cf. Proposition 1.3). Then we show in Proposition 1.4 that the various transformations preserve (regular) holonomicity and (quasi-)coherence. In the last section we introduce in Definition 1.5 the Fourier-Laplace transform for  $\mathcal{D}$ -modules as an integral transform with exponential kernel.

### 1.1 $\mathcal{D}$ -modules

Let  $X$  be a smooth algebraic variety over  $\mathbb{C}$ . We denote by  $M(\mathcal{D}_X)$  the abelian category of algebraic  $\mathcal{D}_X$ -modules on  $X$ . The full triangulated subcategories of  $D^b(\mathcal{D}_X) := D^b(M(\mathcal{D}_X))$ , consisting of objects with (quasi-)coherent resp. (regular) holonomic cohomology are denoted by  $D_{qc}^b(\mathcal{D}_X)$  resp.  $D_{coh}^b(\mathcal{D}_X)$  resp.  $D_{rh}^b(\mathcal{D}_X)$ . We subsume the different cases by writing  $D_*^b(\mathcal{D}_X)$  for  $*$   $\in \{qc, coh, h, rh\}$ .

Let  $f : X \rightarrow Y$  be a map between smooth algebraic varieties. We denote by  $\mathcal{D}_{X \rightarrow Y}$  resp.  $\mathcal{D}_{Y \leftarrow X}$

the transfer bimodules. Let  $M \in D^b(\mathcal{D}_X)$  and  $N \in D^b(\mathcal{D}_Y)$ , then the direct and inverse image for  $\mathcal{D}$ -modules is defined by

$$\begin{aligned} f_+ M &:= Rf_*(\mathcal{D}_{Y \leftarrow X} \overset{L}{\otimes}_{\mathcal{D}_X} M), \\ f^+ N &:= \mathcal{D}_{X \rightarrow Y} \overset{L}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1} M. \end{aligned}$$

Recall that the functors  $f_+, f^+$  preserve quasi-coherence, holonomicity and regular holonomicity (see e.g., [HTT08]).

If  $f : X \rightarrow Y$  is non-characteristic, then the functor  $f^+$  preserves coherency and is exact.

Denote by  $\omega_X$  the canonical line bundle on  $X$ . There exists a duality functor  $\mathbb{D} : D_{coh}^b(\mathcal{D}_X) \rightarrow D_{coh}^b(\mathcal{D}_X)$  defined by

$$\mathbb{D}M := \mathcal{H}om_{\mathcal{O}_X}(\omega_X, R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{D}_X)).$$

Recall that for a single holonomic  $\mathcal{D}_X$ -module  $M$ , the holonomic dual is also a single holonomic  $\mathcal{D}_X$ -module ([HTT08, Corollary 2.6.8 (iii)]).

For a morphism  $f : X \rightarrow Y$  between smooth algebraic varieties we additionally define the functors  $f_{\dagger} := \mathbb{D} \circ f_+ \circ \mathbb{D}$  and  $f^{\dagger} := \mathbb{D} \circ f^+ \circ \mathbb{D}$ .

Consider the following cartesian diagram of algebraic varieties

$$\begin{array}{ccc} Z & \xrightarrow{f'} & W \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

then we have the canonical isomorphism  $f^+ g_+[d] \simeq g'_+ f'^+[d']$ , where  $d := \dim Y - \dim X$  and  $d' := \dim Z - \dim W$  (cf. [HTT08, Theorem 1.7.3]).

Notice that by symmetry we have also the canonical isomorphism  $g^+ f_+[\tilde{d}] \simeq f'_+ g'^+[\tilde{d}']$  with  $\tilde{d} := \dim W - \dim X$  and  $\tilde{d}' := \dim Z - \dim Y$ . In the former case we say we are doing a base change with respect to  $f$ , in the latter case with respect to  $g$ .

Using the duality functor, we get isomorphisms:

$$f^{\dagger} g_{\dagger}[-d] \simeq g'_{\dagger} f'^{\dagger}[-d'] \quad \text{and} \quad g^{\dagger} f_{\dagger}[-\tilde{d}] \simeq f'_{\dagger} g'^{\dagger}[-\tilde{d}'].$$

Let  $M \in D^b(\mathcal{D}_X)$  and  $N \in D^b(\mathcal{D}_Y)$ . We denote by  $M \boxtimes N \in D^b(\mathcal{D}_{X \times Y})$  the exterior tensor product. The exterior tensor product preserves quasi-coherence, coherence, holonomicity and regular holonomicity. If  $M_1, M_2 \in D^b(\mathcal{D}_X)$ , we denote by

$$M_1 \overset{L}{\otimes} M_2 := \Delta^+(M_1 \boxtimes M_2)$$

the internal tensor product, where  $\Delta : X \rightarrow X \times X$  is the diagonal embedding. The internal tensor product preserves quasi-coherence, holonomicity and regular holonomicity. Notice that it preserves

coherency if  $M_1 \boxtimes M_2$  is noncharacteristic with respect to  $\Delta$ , i.e. if  $M_1, M_2$  satisfy the following transversality condition

$$\text{char}(M_1) \cap \text{char}(M_2) \subset T_X^* X.$$

Let  $f : X \rightarrow Y$  a map between smooth algebraic varieties. One has the following projection formula (cf. [HTT08, Corollary 1.7.5])

$$f_+(M \overset{L}{\otimes} f^+ N) \simeq f_+ M \overset{L}{\otimes} N. \quad (1.1.1)$$

## 1.2 Radon transform

Let  $V$  be a complex  $n + 1$ -dimensional vector space and denote by  $\hat{V}$  the dual vector space. Fix  $k \in \{1, \dots, n\}$  and set  $\hat{W} := \hat{V}^k$ , so a point in  $\hat{W}$  gives rise to a vectorsubspace of  $V$  of codimension  $\leq k$ . Set  $d := n - k$  and denote by  $S(k, n)$  the subvariety of  $\hat{W}$  consisting of points giving rise to vectorsubspaces of codimension equal to  $k$  resp.  $d$ -planes in  $\mathbb{P}(V)$ . Notice that  $GL(k)$  acts on  $S(k, n)$  from the left. The quotient

$$G(d, n) := GL(k) \backslash S(k, n)$$

is the Grassmanian parametrizing  $n + 1 - k$ -dimensional subspaces in  $V$ , i.e.  $d$ -planes in  $\mathbb{P}(V)$ . We use the following abbreviations  $\mathbb{P} := \mathbb{P}(V)$ ,  $\dot{V} := V \setminus \{0\}$  and  $\mathbb{G} := G(d, n)$ .

Let  $Z \xrightarrow{i_Z} \mathbb{P} \times \mathbb{G}$  be the universal hyperplane, i.e.

$$Z := \{[v], [\lambda^1, \dots, \lambda^k] \in \mathbb{P} \times \mathbb{G} \mid \lambda^1(v) = \dots = \lambda^k(v) = 0\}$$

and denote by  $U \xrightarrow{j_U} \mathbb{P} \times \mathbb{G}$  the following subvariety

$$U := \{[v], [\lambda^1, \dots, \lambda^k] \in \mathbb{P} \times \mathbb{G} \mid \lambda^1(v) \neq 0, \dots, \lambda^k(v) \neq 0\}.$$

We will define various versions of the Radon transform. Consider the following diagram

$$\begin{array}{ccccc} & & U & & \\ & \swarrow \pi_1^U & \downarrow j_U & \searrow \pi_2^U & \\ \mathbb{P} & \xleftarrow{\pi_1} & \mathbb{P} \times \mathbb{G} & \xrightarrow{\pi_2} & \mathbb{G} \\ & \swarrow \pi_1^Z & \uparrow i_Z & \searrow \pi_2^Z & \\ & & Z & & \end{array},$$

where  $\pi_1$  resp.  $\pi_2$  are the projections to the first resp. second factor and  $\pi_1^U, \pi_1^Z, \pi_2^U, \pi_2^Z$  are the corresponding restrictions of  $\pi_1$  and  $\pi_2$  to  $U$  resp.  $Z$ .

**Definition 1.1.** Let  $V, Z, U$  as above. The Radon transform is the functor

$$\begin{aligned}\mathcal{R}_\delta : D_{qc}^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D^b(\mathcal{D}_{\mathbb{G}}) \\ M &\mapsto \pi_{2+}^Z(\pi_1^Z)^+ M \simeq \pi_{2+} i_{Z+} i_Z^+ \pi_1^+ M.\end{aligned}$$

Define two variants

$$\begin{aligned}\mathcal{R}_{1/t} : D_{qc}^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D^b(\mathcal{D}_{\mathbb{G}}) \\ M &\mapsto \pi_{2+}^U(\pi_1^U)^+ M \simeq \pi_{2+} j_{U+} j_U^+ \pi_1^+ M\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_Y : D_{coh}^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D^b(\mathcal{D}_{\mathbb{G}}) \\ M &\mapsto \pi_{2+}^U(\pi_1^U)^+ M \simeq \pi_{2+} j_{U+} j_U^+ \pi_1^+ M,\end{aligned}$$

as well as a constant Radon transform

$$\begin{aligned}\mathcal{R}_1 : D_{qc}^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D^b(\mathcal{D}_{\mathbb{G}}) \\ M &\mapsto \pi_{2+}(\pi_1)^+ M.\end{aligned}$$

In order to compare the Radon transform to the Fourier-Laplace transform which will be introduced below, we will need another type of Radon transform but this time with target in  $\hat{W} = \hat{V}^{\times k}$  instead of  $\mathbb{G} = G(m-k, m)$ .

Let  $Z' \xrightarrow{i_{Z'}} \mathbb{P} \times \hat{W}$  be defined as

$$Z' := \{[v], \lambda^1, \dots, \lambda^k \in \mathbb{P} \times \hat{W} \mid \lambda^1(v) = \dots = \lambda^k(v) = 0\}$$

and denote by  $U' \xrightarrow{j_{U'}} \mathbb{P} \times \hat{W}$  the following subvariety

$$U' := \{[v], \lambda^1, \dots, \lambda^k \in \mathbb{P} \times \hat{W} \mid \lambda^1(v) \neq 0, \dots, \lambda^k(v) \neq 0\}.$$

Consider the following diagram

$$\begin{array}{ccccc} & & U' & & \\ & \swarrow \pi_1^{U'} & \downarrow j_{U'} & \searrow \pi_2^{U'} & \\ \mathbb{P} & \xleftarrow{\pi_1} & \mathbb{P} \times \hat{W} & \xrightarrow{\pi_2} & \hat{W} \\ & \swarrow \pi_1^{Z'} & \uparrow i_{Z'} & \searrow \pi_2^{Z'} & \\ & & Z' & & \end{array}$$

where as above  $\pi_1$  resp.  $\pi_2$  are the projections to the first resp. second factor and  $\pi_1^{U'}, \pi_1^{Z'}, \pi_2^{U'}, \pi_2^{Z'}$  are the corresponding restrictions of  $\pi_1$  and  $\pi_2$  to  $U'$  resp.  $Z'$ .

**Definition 1.2.** Let  $Z', U'$  as above. The affine Radon transform is the functor

$$\begin{aligned}\mathcal{R}'_\delta : D_{qc}^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D^b(\mathcal{D}_{\hat{W}}) \\ M &\mapsto \pi_{2+}^{Z'}(\pi_1^{Z'})^+ M \simeq \pi_{2+} i_{Z'+} i_{Z'}^+ \pi_1^+ M.\end{aligned}$$

Define two variants

$$\begin{aligned}\mathcal{R}'_{1/t} : D_{qc}^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D^b(\mathcal{D}_{\hat{W}}) \\ M &\mapsto \pi_{2+}^{U'}(\pi_1^{U'})^+ M \simeq \pi_{2+} j_{U'+} j_{U'}^+ \pi_1^+ M\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}'_Y : D_{coh}^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D^b(\mathcal{D}_{\hat{W}}) \\ M &\mapsto \pi_{2+}^{U'}(\pi_1^{U'})^+ M \simeq \pi_{2+} j_{U'+} j_{U'}^+ \pi_1^+ M,\end{aligned}$$

as well as a constant Radon transform

$$\begin{aligned}\mathcal{R}'_1 : D_{qc}^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D^b(\mathcal{D}_{\hat{W}}) \\ M &\mapsto \pi_{2+}(\pi_1)^+ M.\end{aligned}$$

In the following it will sometimes be convenient to use the language of integral kernels. Let  $X$  and  $Y$  be two smooth varieties,  $M \in D^b(\mathcal{D}_X)$  and  $K \in D^b(\mathcal{D}_{X \times Y})$ . Denote by  $q_1 : X \times Y \rightarrow X$  resp.  $q_2 : X \times Y \rightarrow Y$  the projection to the first resp. second factor. The integral transform with respect to the kernel  $K$  is defined by

$$\begin{aligned}\diamond K : D^b(\mathcal{D}_X) &\longrightarrow D^b(\mathcal{D}_Y), \\ M &\mapsto M \diamond K = q_{2+}(q_1^+ M \overset{L}{\otimes} K).\end{aligned}$$

Let  $Z$  be another smooth algebraic variety and  $\tilde{K} \in D^b(\mathcal{D}_{Y \times Z})$ . The convolution of the two kernels  $K$  and  $\tilde{K}$  is define by

$$K \diamond \tilde{K} := q_{13+}(q_{12}^+ K \overset{L}{\otimes} q_{23}^+ \tilde{K}),$$

where  $q_{ij}$  is the projection from  $X \times Y \times Z$  to the corresponding factor. Notice that the folding is associative in the sense that for  $M \in D^b(\mathcal{D}_X)$  we have

$$(M \diamond K) \diamond \tilde{K} \simeq M \diamond (K \diamond \tilde{K}) \in D^b(\mathcal{D}_Z).$$

We can now express the affine Radon transforms  $\mathcal{R}'_u$  for  $u \in \{1, \delta, 1/t, Y\}$  as an integral transformation with respect to a kernel  $R'_u$ . Set

$$R'_Y = j_{U'+} \mathcal{O}_{U'}, \quad R'_{1/t} = j_{U'+} \mathcal{O}_{U'}, \quad R'_\delta = i_{Z'+} \mathcal{O}_{Z'}, \quad R'_1 = \mathcal{O}_{\mathbb{P} \times \hat{W}}.$$

Define

$$M \diamond R'_u := \pi_{2+}(\pi_1^+(M) \overset{L}{\otimes} R'_u),$$

then we have the following result.

**Proposition 1.3.** *Let  $M \in D_{qc}^b(\mathbb{P})$ . We have the following isomorphism*

$$M \diamond R'_u \simeq \mathcal{R}'_u(M) \quad \text{for } u \in \{\delta, 1, 1/t\},$$

*if  $M \in D_{coh}^b(\mathbb{P})$  then*

$$M \diamond R'_Y \simeq \mathcal{R}'_Y(M).$$

*Proof.* The proof essentially uses the projection formula (cf. (1.1.1)). We prove the statement for  $u = \delta$  the other cases are similar or easier.

$$\begin{aligned} M \diamond R'_\delta &= \pi_{2+}(\pi_1^+(M) \overset{L}{\otimes} i_{Z'+} \mathcal{O}_{Z'}) \\ &\simeq \pi_{2+} i_{Z'+} (i_{Z'}^+ \pi_1^+ M \overset{L}{\otimes} \mathcal{O}_{Z'}) \\ &\simeq \pi_{2+}^{Z'} ((\pi_1^{Z'})^+ M) \\ &= \mathcal{R}'_\delta(M) \end{aligned}$$

□

**Proposition 1.4.** *The Radon transforms preserve the following subcategories*

$$\begin{aligned} \mathcal{R}_\delta, \mathcal{R}_{1/t}, \mathcal{R}_1 : D_*^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D_*^b(\mathcal{D}_{\mathbb{G}}) \quad \text{for } * \in \{qc, coh, h, rh\}, \\ \mathcal{R}_Y : D_*^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D_*^b(\mathcal{D}_{\mathbb{G}}) \quad \text{for } * \in \{coh, h, rh\}, \end{aligned}$$

*The affine Radon transforms preserve the following subcategories*

$$\begin{aligned} \mathcal{R}'_\delta, \mathcal{R}'_{1/t}, \mathcal{R}'_1 : D_*^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D_*^b(\mathcal{D}_{\hat{W}}) \quad \text{for } * \in \{qc, coh, h, rh\}, \\ \mathcal{R}'_Y : D_*^b(\mathcal{D}_{\mathbb{P}}) &\rightarrow D_*^b(\mathcal{D}_{\hat{W}}) \quad \text{for } * \in \{coh, h, rh\}, \end{aligned}$$

*Proof.* First notice that the claim is clear for  $* \in \{qc, h, rh\}$ , because the direct and inverse image preserve quasi-coherence, holonomicity and regular holonomicity and the proper direct image preserves holonomicity and regular holonomicity. The functors  $\mathcal{R}_\delta, \mathcal{R}_Y, \mathcal{R}_1$  resp.  $\mathcal{R}'_\delta, \mathcal{R}'_Y, \mathcal{R}'_1$  preserve coherency because  $\pi^U, \pi^Z$  resp.  $\pi^{U'}, \pi^{Z'}$  are smooth and coherency is preserved by proper direct images. In order to prove that  $\mathcal{R}'_{1/t}$  preserves coherency, recall that we have the isomorphism

$$\mathcal{R}'_{1/t}(M) \simeq M \diamond R'_{1/t} \simeq \pi_{2+}(\pi_1^+ M \overset{L}{\otimes} R'_{1/t}).$$

Because  $\pi_2$  is proper, it is enough to show that  $\pi_1^+ M \overset{L}{\otimes} R'_{1/t} \in D_{coh}^b(\mathcal{D}_{\mathbb{P} \times \hat{W}})$  for  $M \in D_{coh}^b(\mathcal{D}_{\mathbb{P}})$ . Notice that  $\text{char}(R'_{1/t}) \subset T_{Z'} \mathbb{P} \times \hat{W}$ . Thus one can easily compute that the transversality condition

$$(\text{char}(M) \times T_{\hat{W}}^* \hat{W}) \cap T_{Z'}(\mathbb{P} \times \hat{W}) \subset T_{\mathbb{P} \times \hat{W}}^* \mathbb{P} \times \hat{W}$$

is satisfied. This shows the claim for  $\mathcal{R}'_{1/t}$ . The proof for  $\mathcal{R}_{1/t}$  can be easily adapted. □

### 1.3 Fourier-Laplace transform

In the next definition we want to introduce the Fourier-Laplace transform.

**Definition 1.5.** Let as above  $V$  be a vectorspace and denote by  $\hat{V}$  its dual. Define  $L := \mathcal{O}_{V \times \hat{V}} e^{-\langle v, \hat{v} \rangle}$  where  $\langle \bullet, \bullet \rangle$  is the natural pairing between  $V$  and  $\hat{V}$  and the  $\mathcal{D}$ -module structure is given by the product rule. Denote by  $\pi_1 : V \times \hat{V} \rightarrow V$ ,  $\pi_2 : V \times \hat{V} \rightarrow \hat{V}$  the canonical projections. The Fourier-Laplace transform is then defined by

$$FL : D_{qc}^b(\mathcal{D}_V) \longrightarrow D_{qc}^b(\mathcal{D}_{\hat{V}}),$$

$$M \mapsto \pi_{2+}(\pi_1^+ M \overset{L}{\otimes}_{\mathcal{O}} L).$$

**Remark 1.6.** In the setting above with  $\hat{W} = \hat{V}^k$  and  $W := V^k$  we can perform the Fourier-Laplace transform stepwise. Set  $N_{j,l} := V^{\times j} \times \hat{V}^{\times l}$  for  $j, l \in \{0, \dots, k\}$  with  $j + l = k + 1$  and denote by  $v^1, \dots, v^j, \lambda^1, \dots, \lambda^l$  elements of  $N_{j,l}$ . Define  $L_{jl} := \mathcal{O}_{N_{j,l}} e^{-\langle v^j, \lambda^1 \rangle}$ . Notice that  $L_{jl} \simeq (\pi^{j,l})^+ L$  if  $\pi^{j,l} : N_{j,l} \rightarrow V \times \hat{V}$  is the canonical projection to the  $j$ -th factor of  $V^{\times j}$  and to the first factor of  $\hat{V}^{\times l}$ . Set

$$FL_j(M) := (\pi_2^{j,l})_+((\pi_1^{j,l})^+ M \otimes L_{jl}),$$

where

$$\begin{aligned} \pi_1^{j,l} : N_{j,l} &\longrightarrow N_{j,l-1}, \\ (v^1, \dots, v^j, \lambda_1, \dots, \lambda^l) &\mapsto (v^1, \dots, v^j, \lambda^2, \dots, \lambda^l), \\ \pi_2^{j,l} : N_{j,l} &\longrightarrow N_{j-1,l}, \\ (v^1, \dots, v^j, \lambda_1, \dots, \lambda^l) &\mapsto (v^1, \dots, v^{j-1}, \lambda^1, \dots, \lambda^l). \end{aligned}$$

One can easily show that

$$FL_1 \circ \dots \circ FL_k(M) \simeq FL(M),$$

where the Fourier-Laplace transform on the right hand side is performed with respect to  $W$ .

## 2 The comparison theorem

In this section we give the proof of the comparison theorem. In the first section we review the theorem of d'Agnolo and Eastwood for the codimension one case. In the second section we state the results for the higher codimension case and in the last section we give the proof of these statements.

### 2.1 The case of codimension one

Denote by  $\tilde{\pi} : \tilde{V} \longrightarrow V$  the blowup of the origin 0 in  $V$  and by  $E$  the exceptional divisor. Then  $\tilde{V}$  carries the following stratification

$$E \xrightarrow{i_E} \tilde{V} \xleftarrow{j_{\tilde{V}}} \dot{V}.$$



Denote by  $\tilde{i} : \tilde{V} \rightarrow \mathbb{P} \times V$  the natural embedding. Consider the maps

$$\mathbb{P} \xleftarrow{\tilde{\pi}} \tilde{V} \xrightarrow{\tilde{j}} V \quad (2.1.1)$$

obtained by restriction of the natural projections from  $\mathbb{P} \times V$  and by

$$\mathbb{P} \xleftarrow{\pi} \dot{V} \xrightarrow{j} V$$

their restriction to  $\dot{V}$ . This gives rise to the following diagram

$$\begin{array}{ccccc} & & \dot{V} & & \\ & \swarrow \pi & \downarrow j_{\dot{V}} & \searrow j & \\ \mathbb{P} & \xleftarrow{\tilde{\pi}} & \tilde{V} & \xrightarrow{\tilde{j}} & V \\ & \swarrow \pi^E & \uparrow i_E & \searrow j_0 & \\ & & E & & \end{array} \quad (2.1.2)$$

We define the following kernels

$$\tilde{S}_1 = \tilde{i}_+ \mathcal{O}_{\tilde{V}_0}, \quad \tilde{S}_Y = \tilde{i}_+ j_{\dot{V}\dagger} \mathcal{O}_{\dot{V}}, \quad \tilde{S}_{1/t} = \tilde{i}_+ j_{\dot{V}+} \mathcal{O}_{\dot{V}}, \quad \tilde{S}_\delta = \tilde{i}_+ i_{E+} \mathcal{O}_E$$

on  $\mathbb{P} \times V$ .

The following result, comparing the affine Radon transform and the Fourier-Laplace transform, is proven in [DE03] for the case  $k = 1$ .

**Proposition 2.1.** [DE03, Proposition 1] *There is the following isomorphism of integral kernels in  $D^b(\mathcal{D}_{\mathbb{P} \times \hat{V}})$ .*

$$\tilde{S}_u \diamond L \simeq R'_u,$$

where

$$u = 1, \delta, Y, 1/t \quad \text{and} \quad \hat{u} = \delta, 1, 1/t, Y.$$

Define the following functors from  $D_{qc}^b(\mathcal{D}_{\mathbb{P}})$  to  $D_{qc}^b(\mathcal{D}_V)$ :

$$\begin{aligned} ext_\delta(M) &:= j_{0+}(\pi^E)^+ M \simeq \tilde{j}_+ i_{E+} i_E^+ \tilde{\pi}^+ M \\ ext_1(M) &:= \tilde{j}_+ \tilde{\pi}^+ M \\ ext_{1/t}(M) &:= j_+ \pi^+ M \simeq \tilde{j}_+ j_{\dot{V}+} j_{\dot{V}}^+ \tilde{\pi}^+(M) \end{aligned}$$

resp. from  $D_{coh}^b(\mathcal{D}_{\mathbb{P}})$  to  $D_{coh}^b(\mathcal{D}_V)$

$$ext_Y(M) := j_{\dagger} \pi^+ M \simeq \tilde{j}_+ j_{\dot{V}\dagger} j_{\dot{V}}^+ \tilde{\pi}^+(M)$$

The proposition above gives the following comparison between the affine Radon transform and the Fourier-Laplace transform.

**Corollary 2.2.** For  $M \in D_{qc}^b(\mathcal{D}_{\mathbb{P}})$  there are natural isomorphisms in  $D_{qc}^b(\mathcal{D}_{\hat{V}})$

$$FL \circ ext_u(M) \simeq \mathcal{R}'_{\hat{u}}(M) \quad u = 1, \delta$$

and for  $M \in D_{coh}^b(\mathcal{D}_{\mathbb{P}})$  there are natural isomorphisms in  $D_{coh}^b(\mathcal{D}_{\hat{V}})$

$$FL \circ ext_u(M) \simeq \mathcal{R}'_{\hat{u}}(M) \quad u = 1/t, Y.$$

*Proof.* We have

$$\mathcal{R}'_{\hat{u}}(M) \simeq M \diamond R'_{\hat{u}} \simeq M \diamond \tilde{S}_u \diamond L \simeq FL(M \diamond \tilde{S}_u).$$

So the only thing to show is  $ext_u(M) \simeq M \diamond \tilde{S}_u$ . But this follows from [DE03, Lemma 1].  $\square$

The two results above which deal with the affine Radon transform were used by d'Agnolo and Eastwood to proof the following theorem, which generalizes a result obtained by Brylinski in [Bry86, Théorème 7.27]

**Theorem 2.3.** [DE03, Theorem 2] For  $M \in D_{qc}^b(\mathcal{D}_{\mathbb{P}})$  there are natural isomorphisms in  $D_{qc}^b(\mathcal{D}_{\hat{V} \setminus \{0\}})$

$$r^+ FL \circ ext_u(M) \simeq \hat{\pi}^+ \mathcal{R}_{\hat{u}}(M) \quad u = 1, \delta$$

and for  $M \in D_{coh}^b(\mathcal{D}_{\mathbb{P}})$  there are natural isomorphisms in  $D_{coh}^b(\mathcal{D}_{\hat{V} \setminus \{0\}})$

$$r^+ FL \circ ext_u(M) \simeq \hat{\pi}^+ \mathcal{R}_{\hat{u}}(M) \quad u = 1/t, Y.$$

where  $r : \hat{V} \setminus \{0\} \rightarrow \hat{V}$  is the natural inclusion and  $\hat{\pi} : \hat{V} \setminus \{0\} \rightarrow \mathbb{P}$  the canonical projection.

## 2.2 Statement of results

In order to state the comparison theorem in the case  $k > 1$  we have to introduce the following maps. Let

$$\Delta : \mathbb{P} \longrightarrow \mathbb{P}^{\times k}$$

be the diagonal embedding. and consider the following diagram which is the  $k$ -th product of diagram (2.1.2)

$$\begin{array}{ccccc} & & \dot{V}^{\times k} & & \\ & \swarrow \pi_k & \downarrow j_{\dot{V},k} & \searrow j_k & \\ \mathbb{P}^{\times k} & \xleftarrow{\tilde{\pi}_k} & \tilde{V}^{\times k} & \xrightarrow{\tilde{j}_k} & V^{\times k} \\ & \swarrow \pi_k^E & \uparrow i_{E,k} & \searrow j_{0,k} & \\ & & E^{\times k} & & \end{array}$$

Define the following functors from  $D_{qc}^b(\mathcal{D}_{\mathbb{P}^{\times k}})$  to  $D_{qc}^b(\mathcal{D}_{V^{\times k}})$ :

$$\begin{aligned} ext_{\delta}^k(M) &:= (j_{0,k})_+(\pi_k^E)^+ M \simeq (\tilde{j}_k)_+(i_{E,k})_+(i_{E,k})^+(\tilde{\pi}_k)^+ M \\ ext_1^k(M) &:= (\tilde{j}_k)_+(\tilde{\pi}_k)^+ M \\ ext_{1/t}^k(M) &:= (j_k)_+(\pi_k)^+ M \simeq (\tilde{j}_k)_+(j_{\dot{V},k})_+(j_{\dot{V},k})^+(\tilde{\pi}_k)^+ M \end{aligned}$$

resp.

$$\text{ext}_Y^k(M) := (j_k)_+(\pi_k)^+ M \simeq (\tilde{j}_k)_+(j_{\hat{V},k})_+(j_{\hat{V},k})^+(\tilde{\pi}_k)^+ M$$

from  $D_{coh}^b(\mathcal{D}_{\mathbb{P}^{\times k}})$  to  $D_{coh}^b(\mathcal{D}_{V^{\times k}})$ .

We have the following generalization of Corollary 2.2

**Proposition 2.4.** *For  $M \in D_{qc}^b(\mathcal{D}_{\mathbb{P}})$  there are natural isomorphisms in  $D_{qc}^b(\mathcal{D}_{\hat{W}})$*

$$\text{FL} \circ \text{ext}_u^k \circ \Delta_+(M) \simeq \mathcal{R}'_{\hat{u}}(M) \quad u = 1, \delta$$

and for  $M \in D_{coh}^b(\mathcal{D}_{\mathbb{P}})$  there are natural isomorphisms in  $D_{coh}^b(\mathcal{D}_{\hat{W}})$

$$\text{FL} \circ \text{ext}_u^k \circ \Delta_+(M) \simeq \mathcal{R}'_{\hat{u}}(M) \quad u = 1/t, Y.$$

The corresponding generalization of Theorem 2.3 is the following theorem.

**Theorem 2.5.** *For  $M \in D_{qc}^b(\mathcal{D}_{\mathbb{P}})$  there are natural isomorphisms in  $D_{qc}^b(\mathcal{D}_{S(k,n)})$*

$$r^+(\text{FL} \circ \text{ext}_u^k \circ \Delta_+(M)) \simeq \hat{\pi}^+ \mathcal{R}_{\hat{u}}(M) \quad u = 1, \delta$$

and for  $M \in D_{coh}^b(\mathcal{D}_{\mathbb{P}})$  there are natural isomorphisms in  $D_{coh}^b(\mathcal{D}_{S(k,n)})$

$$r^+(\text{FL} \circ \text{ext}_u^k(M) \circ \Delta_+(M)) \simeq \hat{\pi}^+ \mathcal{R}_{\hat{u}}(M) \quad u = 1/t, Y.$$

where  $r : S(k, n) \rightarrow \hat{W} = \hat{V}^{\times k}$  is the natural inclusion and  $\hat{\pi} : S(k, n) \rightarrow \mathbb{G}$  the canonical projection.

## 2.3 Proof of the higher codimension case

In this section we give the proof of the comparison theorem in the case of  $k > 1$ , using the case  $k = 1$  already proven by d'Agnolo and Eastwood.

We first introduce various Radon like transformations on the product space  $\mathbb{P}^{\times k}$  and prove in Lemma 2.6 that these transformations applied to some  $\mathcal{D}$ -module  $M$  which is supported on the diagonal of  $\mathbb{P}^{\times k}$  are isomorphic to the  $d$ -plane Radon transforms of  $M$ . Set

$$\tilde{Z} := \{([v^1], \dots, [v^k], \lambda^1, \dots, \lambda^k) \in \prod_{i=1}^k \mathbb{P} \times \hat{W} \mid \lambda^i(v^i) = 0 \ \forall i \in \{1, \dots, k\}\}$$

and denote by  $i_{\tilde{Z}} : \tilde{Z} \rightarrow \mathbb{P}^{\times k} \times \hat{W}$  its inclusion. Define the open subvariety

$$\tilde{U} := \{([v^1], \dots, [v^k], \lambda^1, \dots, \lambda^k \in \mathbb{P}^{\times k} \times \hat{W} \mid \lambda^1(v^1) \neq 0, \dots, \lambda^k(v^k) \neq 0\}$$

and let  $j_{\tilde{U}} : \tilde{U} \rightarrow \mathbb{P}^{\times k} \times \hat{W}$  be the canonical inclusion. Denote by  $\pi_1^{\Delta}, \pi_2^{\Delta}$  the projections from  $\mathbb{P}^{\times k} \times \hat{W}$  to  $\mathbb{P}^{\times k}$  resp.  $\hat{W}$  and denote their restrictions to  $\tilde{Z}$  resp.  $\tilde{U}$  by  $\pi_1^{\tilde{Z}}, \pi_2^{\tilde{Z}}$  resp.  $\pi_1^{\tilde{U}}, \pi_2^{\tilde{U}}$ .

We define the following transformations for  $M \in D_{qc}^b(\mathcal{D}_{\mathbb{P} \times k})$

$$\begin{aligned}\tilde{\mathcal{R}}_\delta(M) &:= \pi_{2+}^{\tilde{Z}}(\pi_1^{\tilde{Z}})^+ M \simeq \pi_{2+}^\Delta i_{\tilde{Z}+} i_{\tilde{Z}}^+(\pi_1^\Delta)^+ M, \\ \tilde{\mathcal{R}}_{1/t}(M) &:= \pi_{2+}^{\tilde{U}}(\pi_1^{\tilde{U}})^+ M \simeq \pi_{2+}^\Delta j_{\tilde{U}+} j_{\tilde{U}}^+(\pi_1^\Delta)^+ M, \\ \tilde{\mathcal{R}}_1(M) &:= \pi_{2+}^\Delta (\pi_1^\Delta)^+ M\end{aligned}$$

and for  $M \in D_{coh}^b(\mathcal{D}_{\mathbb{P} \times k})$

$$\tilde{\mathcal{R}}_Y(M) := \pi_{2+}^{\tilde{U}}(\pi_1^{\tilde{U}})^+ M \simeq \pi_{2+}^\Delta j_{\tilde{U}+} j_{\tilde{U}}^+(\pi_1^\Delta)^+ M.$$

Consider the following diagram

$$\begin{array}{ccccc} U' & \xrightarrow{j_{U'}} & \mathbb{P} \times \hat{W} & \xleftarrow{i_{Z'}} & Z' \\ \Delta_{U'} \downarrow & \nearrow \pi_1 & \downarrow \Delta \times id_{\hat{W}} & \nearrow \pi_2^\Delta & \downarrow \Delta_{Z'} \\ \tilde{U} & \xrightarrow{j_{\tilde{U}}} & \mathbb{P} \times k \times \hat{W} & \xleftarrow{i_{\tilde{Z}}} & \tilde{Z} \\ & \downarrow \Delta & \nearrow \pi_1^\Delta & & \\ & \mathbb{P} \times k & & & \end{array}$$

**Lemma 2.6.** *We have the following isomorphism for  $M \in D_{qc}^b(\mathcal{D}_{\mathbb{P}})$*

$$\mathcal{R}'_{\hat{u}}(M) \simeq \tilde{\mathcal{R}}_{\hat{u}}(\Delta_+(M)) \quad \hat{u} = \delta, 1, 1/t$$

and for  $M \in D_{coh}^b(\mathcal{D}_{\mathbb{P}})$

$$\mathcal{R}'_Y(M) \simeq \tilde{\mathcal{R}}_Y(\Delta_+(M)).$$

*Proof.* We are going to show  $\mathcal{R}'_\delta(M) \simeq \tilde{\mathcal{R}}_\delta(\Delta_+(M))$  the other cases are similar or even simpler.

$$\begin{aligned}\mathcal{R}'_\delta(M) &\simeq \pi_{2+} i_{Z'+} i_{Z'}^+ \pi_1^+ M \\ &\simeq \pi_{2+}^\Delta (\Delta \times id_w)_+ i_{Z'+} i_{Z'}^+ \pi_1^+ M \\ &\simeq \pi_{2+}^\Delta i_{\tilde{Z}+} \Delta_{Z'+} i_{Z'}^+ \pi_1^+ M \\ &\simeq \pi_{2+}^\Delta i_{\tilde{Z}+} i_{\tilde{Z}}^+ (\Delta \times id_W)_+ \pi_1^+ M \\ &\simeq \pi_{2+}^\Delta i_{\tilde{Z}+} i_{\tilde{Z}}^+ \pi_{1+}^\Delta \Delta_+ M \\ &= \tilde{\mathcal{R}}_\delta(\Delta_+(M)).\end{aligned}$$

□

Now we want to define transformations  $\tilde{\mathcal{R}}_{\hat{u}}^i$  which can be considered as partial Radon transforms with respect to the  $i$ -th factor of  $\mathbb{P}^{\times k}$ . We will prove in Lemma 2.8 that the Radon like transformations  $\tilde{\mathcal{R}}_{\hat{u}}$  introduced above are actually equivalent to the consecutive application of the corresponding hyperplane Radon transforms  $\tilde{\mathcal{R}}_{\hat{u}}^i$  on each factor.

Define  $T_{i,j} := \mathbb{P}^{\times i} \times \hat{V}^{\times j}$  for  $i, j \in \{0, \dots, k\}$  with  $k \leq i+j \leq 2 \cdot k$  and consider the following diagram of spaces

$$\begin{array}{ccccc}
T_{k,k} & \cdots & T_{1,k} & \xrightarrow[\pi_2^{T_{1,k}}]{\pi_1^{T_{1,k}}} & T_{0,k} \\
& & & \downarrow & \\
& & & T_{1,k-1} & \\
& & \cdots & & \\
& & T_{k-1,2} & \xrightarrow[\pi_2^{T_{k-1,2}}]{\pi_1^{T_{k-1,2}}} & T_{k-2,2} \\
& \swarrow & & & \\
T_{k,2} & \xrightarrow[\pi_2^{T_{k,2}}]{\pi_1^{T_{k,2}}} & T_{k-1,2} & & \\
\downarrow & & \downarrow & & \\
T_{k,1} & \xrightarrow[\pi_2^{T_{k,1}}]{\pi_1^{T_{k,1}}} & T_{k-1,1} & & \\
\downarrow & & & & \\
T_{k,0} & & & & 
\end{array}$$

where the maps are defined as follows:

$$\begin{aligned}
\pi_1^{T_{i,j}} : T_{i,j} &\longrightarrow T_{i,j-1}, \\
([v^1], \dots, [v^i], \lambda^1, \dots, \lambda^j) &\mapsto ([v^1], \dots, [v^i], \lambda^2, \dots, \lambda^j), \\
\pi_2^{T_{i,j}} : T_{i,j} &\longrightarrow T_{i-1,j}, \\
([v^1], \dots, [v^i], \lambda^1, \dots, \lambda^j) &\mapsto ([v^1], \dots, [v^{i-1}], \lambda^1, \dots, \lambda^j)
\end{aligned}$$

for  $i, j \in \{1, \dots, k\}$  with  $k+1 \leq i+j \leq 2 \cdot k$ . Now it is easy to see that the squares

$$\begin{array}{ccc}
T_{i,j} & \xrightarrow[\pi_2^{T_{i,j}}]{\pi_1^{T_{i,j}}} & T_{i-1,j} \\
\downarrow \pi_1^{T_{i,j}} & & \downarrow \pi_1^{T_{i-1,j}} \\
T_{i,j-1} & \xrightarrow[\pi_2^{T_{i,j-1}}]{\pi_1^{T_{i,j-1}}} & T_{i-1,j-1}
\end{array}$$

are cartesian for  $i, j \in \{2, \dots, k\}$  with  $k+2 \leq i+j \leq 2 \cdot k$ .

We now define closed subvarieties  $Z_{i,j} \subset T_{i,j} = \mathbb{P}^{\times i} \times \hat{V}^{\times j}$  for  $i, j \in \{0, \dots, k\}$  with  $k+1 \leq i+j \leq 2 \cdot k$ :

$$Z_{i,j} := \{([v^1], \dots, [v^i], \lambda^1, \dots, \lambda^j) \in T_{i,j} \mid \lambda^1(v^{k-j+1}) = \dots = \lambda^{i+j-k}(v^i) = 0\}.$$

Notice that

$$Z_{k,k} = \{[v^1], \dots, [v^k], \lambda^1, \dots, \lambda^k \in T_{k,k} \mid \lambda^1(v^1) = \dots \lambda^k(v^k) = 0\} = \tilde{Z}$$

and

$$Z_{i,k-i+1} = \{[v^1], \dots, [v^i], \lambda^1, \dots, \lambda^{k-i+1} \in T_{i,k-i+1} \mid \lambda^1(v^i) = 0\}$$

for  $i \in \{1, \dots, k\}$ .

**Lemma 2.7.** *Let  $\pi_1^{Z_{i,j}}$  resp.  $\pi_2^{Z_{i,j}}$  be the restrictions of  $\pi_1^{T_{i,j}}$  resp.  $\pi_2^{T_{i,j}}$  to  $Z_{i,j}$ . The squares*

$$\begin{array}{ccc} Z_{i,j} & \xrightarrow{\pi_2^{Z_{i,j}}} & Z_{i-1,j} \\ \pi_1^{Z_{i,j}} \downarrow & & \downarrow \pi_1^{Z_{i-1,j}} \\ Z_{i,j-1} & \xrightarrow{\pi_2^{Z_{i,j-1}}} & Z_{i-1,j-1} \end{array} \quad \text{and} \quad \begin{array}{ccc} Z_{r+1,s+1} & \xrightarrow{\pi_2^{Z_{r+1,s+1}}} & Z_{r,s+1} \\ \pi_1^{Z_{r+1,s+1}} \downarrow & & \downarrow \pi_1^{Z_{r,s+1}} \\ Z_{r+1,s} & \xrightarrow{\pi_2^{Z_{r+1,s}}} & T_{r,s} \end{array}$$

are cartesian for  $i, j \in \{2, \dots, k\}$  with  $k+3 \leq i+j \leq 2 \cdot k$  and for  $r, s \in \{1, \dots, k-1\}$  with  $r+s = k$ .

*Proof.* First, we write down the definition of the spaces involved

$$\begin{aligned} Z_{i,j} &= \{([v^1], \dots, [v^i], \lambda^1, \dots, \lambda^j) \in T_{i,j} \mid \lambda^1(v^{k-j+1}) = \dots = \lambda^{i+j-k}(v^i) = 0\}, \\ Z_{i-1,j} &= \{([v^1], \dots, [v^{i-1}], \lambda^1, \dots, \lambda^j) \in T_{i-1,j} \mid \lambda^1(v^{k-j+1}) = \dots = \lambda^{i-1+j-k}(v^{i-1}) = 0\}, \\ Z_{i,j-1} &= \{([v^1], \dots, [v^i], \lambda^2, \dots, \lambda^j) \in T_{i,j-1} \mid \lambda^2(v^{k-j+2}) = \dots = \lambda^{i+j-k}(v^i) = 0\}, \\ Z_{i-1,j-1} &= \{([v^1], \dots, [v^{i-1}], \lambda^2, \dots, \lambda^{j-1}) \in T_{i-1,j-1} \mid \lambda^2(v^{k-j+2}) = \dots = \lambda^{i-1+j-k}(v^{i-1}) = 0\}, \end{aligned}$$

where we changed the indices of the elements in the definition of the spaces  $Z_{i,j-1}$  and  $Z_{i-1,j-1}$ . Now one can see rather easily that the maps

$$\begin{array}{ccc} Z_{i,j} & \xrightarrow{\pi_2^{Z_{i,j}}} & Z_{i-1,j} \\ \pi_1^{Z_{i,j}} \downarrow & & \\ Z_{i,j-1} & & \end{array}$$

are well-defined and that the left square is cartesian. The proof of the fact that the right square is cartesian is completely parallel if one sets  $Z_{i,j} := T_{i,j}$  for  $i+j = k$ .  $\square$

The lemma above gives rise to the following diagram of cartesian squares

$$\begin{array}{ccccc}
 Z_{k,k} & \cdots & Z_{1,k} & \xrightarrow{\pi_2^{Z_{1,k}}} & T_{0,k} \\
 \vdots & & \downarrow \pi_1^{Z_{1,k}} & & \downarrow \pi_2^{Z_{1,k}} \\
 & & T_{1,k-1} & & \\
 & & \nearrow & & \\
 Z_{k,2} & \xrightarrow{\pi_2^{Z_{k,2}}} & Z_{k-1,2} & \xrightarrow{\pi_2^{Z_{k-1,2}}} & T_{k-2,2} \\
 \downarrow \pi_1^{Z_{k,2}} & & \downarrow \pi_1^{Z_{k-1,2}} & & \\
 Z_{k,1} & \xrightarrow{\pi_2^{Z_{k,1}}} & T_{k-1,1} & & \\
 \downarrow \pi_1^{Z_{k,1}} & & & & \\
 T_{k,0} & & & & 
 \end{array} \tag{2.3.1}$$

We now define open subvarieties  $U_{i,j} \subset T_{i,j} = \mathbb{P}^{\times i} \times \hat{V}^{\times j}$  for  $i, j \in \{0, \dots, k\}$  with  $k+1 \leq i+j \leq 2 \cdot k$ :

$$U_{i,j} := \{([v^1], \dots, [v^i], \lambda^1, \dots, \lambda^j) \in T_{i,j} \mid \lambda^1(v^{k-j+1}) \neq 0, \dots, \lambda^{i+j-k}(v^i) \neq 0\}.$$

Notice that

$$U_{k,k} = \{[v^1], \dots, [v^k], \lambda^1, \dots, \lambda^k \in T_{k,k} \mid \lambda^1(v^1) \neq 0, \dots, \lambda^k(v^k) \neq 0\} = \tilde{U}$$

and

$$U_{i,k-i+1} = \{[v^1], \dots, [v^i], \lambda^1, \dots, \lambda^{k-i+1} \in T_{i,k-i+1} \mid \lambda^1(v^i) \neq 0\}$$

for  $i \in \{1, \dots, k\}$ . Arguing as in the case of the  $Z_{i,j}$  we get a diagram of cartesian squares

$$\begin{array}{ccccc}
 U_{k,k} & \cdots & U_{1,k} & \xrightarrow{\pi_2^{U_{1,k}}} & T_{0,k} \\
 \vdots & & \downarrow \pi_1^{U_{1,k}} & & \downarrow \pi_2^{U_{1,k}} \\
 & & T_{1,k-1} & & \\
 & & \nearrow & & \\
 U_{k,2} & \xrightarrow{\pi_2^{U_{k,2}}} & U_{k-1,2} & \xrightarrow{\pi_2^{U_{k-1,2}}} & T_{k-2,2} \\
 \downarrow \pi_1^{U_{k,2}} & & \downarrow \pi_1^{U_{k-1,2}} & & \\
 U_{k,1} & \xrightarrow{\pi_2^{U_{k,1}}} & T_{k-1,1} & & \\
 \downarrow \pi_1^{U_{k,1}} & & & & \\
 T_{k,0} & & & & 
 \end{array}$$

We have the following diagram

$$\begin{array}{ccccc}
 & & U_{i,j} & & \\
 & \swarrow \pi_1^{U_{i,j}} & \downarrow j_{U_{i,j}} & \searrow \pi_2^{U_{i,j}} & \\
 T_{i,j-1} & \xleftarrow{\pi_1^{T_{i,j}}} & T_{i,j} & \xrightarrow{\pi_2^{T_{i,j}}} & T_{i-1,j} \\
 & \swarrow \pi_1^{Z_{i,j}} & \downarrow i_{Z_{i,j}} & \searrow \pi_2^{Z_{i,j}} & \\
 & & Z_{i,j} & & 
 \end{array} \tag{2.3.2}$$

for  $i, j \in \{0, \dots, k\}$  with  $i + j = k + 1$ . Now define the following partial Radon transforms for  $M \in D_{qc}^b(\mathcal{D}_{T_{i,j-1}})$

$$\begin{aligned}
 \tilde{\mathcal{R}}_\delta^i(M) &:= \pi_{2+}^{Z_{i,j}}(\pi_1^{Z_{i,j}})^+ M \simeq \pi_{2+}^{T_{i,j}} i_{Z_{i,j}}^+ i_{Z_{i,j}}^+ (\pi_1^{T_{i,j}})^+ M, \\
 \tilde{\mathcal{R}}_{1/t}^i(M) &:= \pi_{2+}^{U_{i,j}}(\pi_1^{U_{i,j}})^+ M \simeq \pi_{2+}^{T_{i,j}} j_{U_{i,j}}^+ j_{U_{i,j}}^+ (\pi_1^{T_{i,j}})^+ M, \\
 \tilde{\mathcal{R}}_1^i(M) &:= \pi_{2+}^{T_{i,j}}(\pi_1^{T_{i,j}})^+ M
 \end{aligned}$$

and for  $M \in D_{coh}^b(\mathcal{D}_{T_{i,j-1}})$

$$\tilde{\mathcal{R}}_Y^i(M) := \pi_{2+}^{U_{i,j}}(\pi_1^{U_{i,j}})^+ M \simeq \pi_{2+}^{T_{i,j}} j_{U_{i,j}}^+ j_{U_{i,j}}^+ (\pi_1^{T_{i,j}})^+ M.$$

**Lemma 2.8.** *We have the following isomorphisms for  $M \in D_{qc}^b(\mathcal{D}_{T_{i,j-1}})$*

$$\tilde{\mathcal{R}}_{\hat{u}}^1 \circ \dots \circ \tilde{\mathcal{R}}_{\hat{u}}^k(M) \simeq \tilde{\mathcal{R}}_{\hat{u}}(M) \quad \hat{u} = 1, \delta, 1/t$$

and for  $M \in D_{coh}^b(\mathcal{D}_{T_{i,j-1}})$

$$\tilde{\mathcal{R}}_Y^1 \circ \dots \circ \tilde{\mathcal{R}}_Y^k(M) \simeq \tilde{\mathcal{R}}_Y(M).$$

*Proof.* We will only show  $\tilde{\mathcal{R}}_\delta^1 \circ \dots \circ \tilde{\mathcal{R}}_\delta^k(M) \simeq \tilde{\mathcal{R}}_\delta(M)$  the other cases are again similar or simpler. Recall the pyramid diagram 2.3.1 from above. The squares

$$\begin{array}{ccc}
 Z_{i,j} & \xrightarrow{\pi_2^{Z_{i,j}}} & Z_{i-1,j} \\
 \pi_1^{Z_{i,j}} \downarrow & & \downarrow \pi_1^{Z_{i-1,j}} \\
 Z_{i,j-1} & \xrightarrow{\pi_2^{Z_{i,j-1}}} & Z_{i-1,j-1}
 \end{array} \quad \text{and} \quad \begin{array}{ccc}
 Z_{r+1,s+1} & \xrightarrow{\pi_2^{Z_{r+1,s+1}}} & Z_{r,s+1} \\
 \pi_1^{Z_{r+1,s+1}} \downarrow & & \downarrow \pi_1^{Z_{r,s+1}} \\
 Z_{r+1,s} & \xrightarrow{\pi_2^{Z_{r+1,s}}} & T_{r,s}
 \end{array}$$

are cartesian for  $i, j \in \{2, \dots, k\}$  with  $k+3 \leq i+j \leq 2 \cdot k$  and for  $r, s \in \{1, \dots, k-1\}$  with  $r+s = k$ .

By base change we have the following isomorphisms of functors

$$(\pi_1^{Z_{i-1,j}})^+(\pi_2^{Z_{i,j-1}})^+ \simeq (\pi_2^{Z_{i,j}})^+(\pi_1^{Z_{i,j}})^+ \quad \text{and} \quad (\pi_1^{Z_{r,s+1}})^+(\pi_2^{Z_{r+1,s}})^+ \simeq (\pi_2^{Z_{r+1,s+1}})^+(\pi_1^{Z_{r+1,s+1}})^+.$$



Thus we have

$$\begin{aligned}\tilde{\mathcal{R}}_\delta^1 \circ \dots \circ \tilde{\mathcal{R}}_\delta^k(M) &= (\pi_2^{Z_{1,k}})_+ (\pi_1^{Z_{1,k}})^+ \dots (\pi_2^{Z_{k,1}})_+ (\pi_1^{Z_{k,1}})^+ M \\ &\simeq (\pi_2^{Z_{1,k}})_+ \dots (\pi_2^{Z_{k-1,k}})_+ (\pi_2^{Z_{k,k}})_+ (\pi_1^{Z_{k,k}})^+ \dots (\pi_1^{Z_{k,2}})^+ (\pi_1^{Z_{k,1}})^+ M.\end{aligned}$$

It now follows from the construction of the  $Z_{i,j}$  that  $Z_{k,k} \simeq \tilde{Z}$  and, if we identify the latter spaces, that

$$\pi_2^{Z_{1,k}} \dots \pi_2^{Z_{k-1,k}} \pi_2^{Z_{k,k}} = \pi_2^{\tilde{Z}} \quad \text{and} \quad \pi_1^{Z_{1,k}} \dots \pi_1^{Z_{k,k-1}} \pi_1^{Z_{k,k}} = \pi_1^{\tilde{Z}}.$$

Thus

$$\begin{aligned}\tilde{\mathcal{R}}_\delta^1 \circ \dots \circ \tilde{\mathcal{R}}_\delta^k(M) &\simeq \pi_{2+}^{\tilde{Z}} (\pi_1^{\tilde{Z}})^+ M \\ &\simeq \tilde{\mathcal{R}}_\delta(M).\end{aligned}$$

□

As a next step we want to compare the partial Radon transform  $\tilde{\mathcal{R}}_u^i$  with a partial Fourier-Laplace transform. For this, we need a description of the partial Radon transform as an integral transformation with kernel  $\tilde{R}_u^i$ . Recall diagram (2.3.2) and set

$$\tilde{R}_Y^i := j_{U_{i,j}}^\dagger \mathcal{O}_{U_{i,j}}, \quad \tilde{R}_{1/t}^i := j_{U_{i,j}} \mathcal{O}_{U_{i,j}}, \quad \tilde{R}_\delta^i := i_{Z_{i,j}} \mathcal{O}_{Z_{i,j}}, \quad \tilde{R}_1^i := \mathcal{O}_{T_{i,j}}.$$

for  $i \in \{1, \dots, k\}$  and  $j = k + 1 - i$ . By arguing as in Proposition 1.3, we get

$$M \diamond \tilde{R}_u^i := \pi_{2+}^{T_{i,j}} ((\pi_1^{T_{i,j}})^+ (M) \otimes^L \tilde{R}_u^i) \simeq \tilde{\mathcal{R}}_u^i(M)$$

for  $M \in D_{qc}^b(\mathcal{D}_{T_{i,j-1}})$  and  $u = \delta, 1, 1/t$  resp.  $M \in D_{coh}^b(\mathcal{D}_{T_{i,j-1}})$  and  $u = Y$ .

Recall the definition of the space  $T_{i,j}$  and define for  $i + j = k + 1$ :

$$T_i := T_{i,j} = \prod_{l=1}^{i-1} \mathbb{P} \times (\mathbb{P} \times \hat{V}) \times \prod_{l=i}^{k-1} \hat{V}$$

together with the following projection:

$$\begin{aligned}\pi^{T_i} : T_i &\longrightarrow \mathbb{P} \times \hat{V}, \\ ([v^1], \dots, [v^{i-1}], [v^i], \lambda^1, \lambda^2, \dots, \lambda^{k+1-i}) &\mapsto ([v^i], \lambda_1).\end{aligned}$$

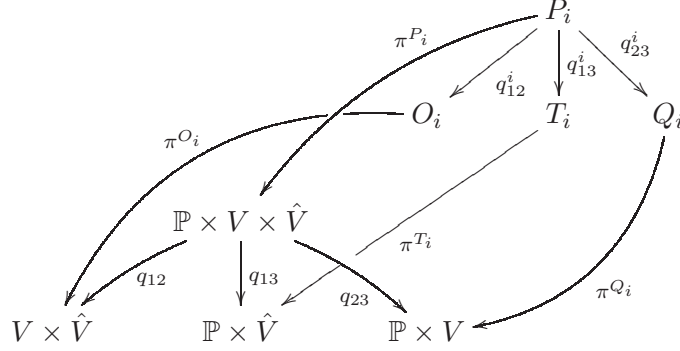
We also need to define the following spaces

$$\begin{aligned}Q_i &:= \prod_{l=1}^{i-1} \mathbb{P} \times (\mathbb{P} \times V) \times \prod_{l=i}^{k-1} \hat{V}, \\ O_i &:= \prod_{l=1}^{i-1} \mathbb{P} \times (V \times \hat{V}) \times \prod_{l=i}^{k-1} \hat{V}, \\ P_i &:= \prod_{l=1}^{i-1} \mathbb{P} \times (\mathbb{P} \times V \times \hat{V}) \times \prod_{l=i}^{k-1} \hat{V},\end{aligned}$$

together with the corresponding projections

$$\begin{aligned}\pi^{Q_i} : Q_i &\longrightarrow \mathbb{P} \times V, \\ \pi^{O_i} : O_i &\longrightarrow V \times \hat{V}, \\ \pi^{P_i} : P_i &\longrightarrow \mathbb{P} \times V \times \hat{V}.\end{aligned}$$

This gives rise to the following commutative diagram



We consider now the following lifts of the kernels  $\tilde{i}_+ \tilde{S}_u$  on  $\mathbb{P} \times V$  to  $T_i$ :

$$\tilde{S}_1^i := (\pi^{Q_i})^+ \tilde{i}_+ \mathcal{O}_{\tilde{V}}, \quad \tilde{S}_Y^i := (\pi^{Q_i})^+ \tilde{i}_+ (j_{V+} \mathcal{O}_{\tilde{V}}), \quad \tilde{S}_{1/t}^i := (\pi^{Q_i})^+ \tilde{i}_+ (j_{V+} \mathcal{O}_{\tilde{V}}), \quad \tilde{S}_\delta^i := (\pi^{Q_i})^+ \tilde{i}_+ (i_{E+} \mathcal{O}_E)$$

and

$$L_i := \pi^{O_i} + L$$

on  $O_i$ . Notice that we have  $\tilde{R}_u^i \simeq \pi^{T_i} + R_u^i$ .

**Proposition 2.9.** *We have the following isomorphisms of kernels in  $D^b(\mathbb{P} \times \hat{V})$ :*

$$\tilde{S}_u^i \diamond L_i \simeq \tilde{R}_u^i.$$

*Proof.*

$$\begin{aligned}\tilde{S}_u^i \diamond L_i &\simeq q_{13+}^i ((q_{23}^i)^+ \tilde{S}_u^i \overset{L}{\otimes} (q_{12}^i)^+ L_i) \\ &\simeq q_{13+}^i ((q_{23}^i)^+ (\pi^{Q_i})^+ \tilde{S}_u^i \overset{L}{\otimes} (q_{12}^i)^+ (\pi^{O_i})^+ L) \\ &\simeq q_{13+}^i ((\pi^{P_i})^+ q_{23+}^+ \tilde{S}_u^i \overset{L}{\otimes} (\pi^{P_i})^+ q_{12+}^+ L) \\ &\simeq q_{13+}^i (\pi^{P_i})^+ (q_{23+}^+ \tilde{S}_u^i \overset{L}{\otimes} q_{12+}^+ L) \\ &\simeq (\pi^{T_i})^+ q_{13+} (q_{23+}^+ \tilde{S}_u^i \overset{L}{\otimes} q_{12+}^+ L) \\ &\simeq (\pi^{T_i})^+ (\tilde{S}_u^i \diamond L) \\ &\simeq (\pi^{T_i})^+ (R_u^i) \\ &\simeq R_u^i,\end{aligned}\tag{2.3.3}$$

where (2.3.3) follows from Proposition 2.1. □

Now define the following spaces

$$N_{i,j,l} := \mathbb{P}^{\times i} \times V^{\times j} \times \hat{V}^{\times l}.$$

with  $i, j, l \in \mathbb{N}_0$  and  $i + j + l = k$ . Notice that  $T_{k-i,i} = N_{k-i,0,i}$ . We define functors

$$ext_u^{ijl} : D_{qc}^b(\mathcal{D}_{N_{i,j,l}}) \rightarrow D_{qc}^b(\mathcal{D}_{N_{i-1,j+1,l}}) \quad \text{for } u = \delta, 1, 1/t$$

and

$$ext_Y^{ijl} : D_{coh}^b(\mathcal{D}_{N_{i,j,l}}) \rightarrow D_{coh}^b(\mathcal{D}_{N_{i-1,j+1,l}}),$$

which are lifts of the functors

$$ext_u : D_{qc}^b(\mathcal{D}_{\mathbb{P}}) \longrightarrow D_{qc}^b(\mathcal{D}_V) \quad \text{for } u = \delta, 1, 1/t,$$

$$M \mapsto ext_u(M) \simeq \pi_{2+}(\pi_1^+ M \overset{L}{\otimes} \tilde{S}_u)$$

resp.  $ext_Y : D_{coh}^b(\mathcal{D}_{\mathbb{P}}) \rightarrow D_{coh}^b(\mathcal{D}_{\mathbb{P}})$ . More precisely, let

$$\pi_1^{ijl} : \mathbb{P}^{\times(i-1)} \times (\mathbb{P} \times V) \times V^{\times j} \times \hat{V}^{\times l} \rightarrow N_{i,j,l} = \mathbb{P}^{\times(i-1)} \times \mathbb{P} \times V^{\times j} \times \hat{V}^{\times l}, \quad (2.3.4)$$

$$\pi_2^{ijl} : \mathbb{P}^{\times(i-1)} \times (\mathbb{P} \times V) \times V^{\times j} \times \hat{V}^{\times l} \rightarrow N_{i-1,j+1,l} = \mathbb{P}^{\times(i-1)} \times V \times V^{\times j} \times \hat{V}^{\times l}, \quad (2.3.5)$$

$$\pi^{ijl} : \mathbb{P}^{\times(i-1)} \times (\mathbb{P} \times V) \times V^{\times j} \times \hat{V}^{\times l} \rightarrow \mathbb{P} \times V \quad (2.3.6)$$

be the canonical projections, then

$$ext_u^{ijl}(M) := (\pi_2^{ijl})_+ \left( (\pi_1^{ijl})^+ M \overset{L}{\otimes} (\pi^{ijl})^+ \tilde{S}_u \right)$$

for  $M \in D_{qc}^b(\mathcal{D}_{N_{i,j,l}})$  and  $u = \delta, 1, 1/t$  resp.  $M \in D_{coh}^b(\mathcal{D}_{N_{i,j,l}})$  and  $u = Y$ .

We also want to define various partial Fourier-Laplace transforms

$$FL_{ijl} : D_{qc}^b(\mathcal{D}_{N_{i,j,l}}) \longrightarrow D_{qc}^b(\mathcal{D}_{N_{i,j-1,l+1}}),$$

which lift the corresponding functor

$$FL : D_{qc}^b(\mathcal{D}_V) \longrightarrow D_{qc}^b(\mathcal{D}_{\hat{V}}), \\ M \mapsto FL(M),$$

i.e. let

$$\begin{aligned} \chi_1^{ijl} : \mathbb{P}^{\times i} \times V^{\times(j-1)} \times (V \times \hat{V}) \times \hat{V}^{\times l} &\longrightarrow N_{i,j,l} = \mathbb{P}^{\times i} \times V^{\times(j-1)} \times V \times \hat{V}^{\times l}, \\ \chi_2^{ijl} : \mathbb{P}^{\times i} \times V^{\times(j-1)} \times (V \times \hat{V}) \times \hat{V}^{\times l} &\longrightarrow N_{i,j-1,l+1} = \mathbb{P}^{\times i} \times V^{\times(j-1)} \times \hat{V} \times \hat{V}^{\times l}, \\ \chi^{ijl} : \mathbb{P}^{\times i} \times V^{\times(j-1)} \times (V \times \hat{V}) \times \hat{V}^{\times l} &\longrightarrow V \times \hat{V} \end{aligned}$$

be the canonical projections, then

$$FL_{ijl}(M) := (\chi_2^{ijl})_+ \left( (\chi_1^{ijl})^+ M \otimes^L (\chi^{ijl})^+ L \right).$$

Now consider the following diagram of functors (for readability we just wrote the spaces and not the categories on which the functors are defined)

$$\begin{array}{ccccccc}
& & & & & N_{0,0,k} & \\
& & & & & \nearrow \mathcal{R}_u^1 & \\
& & & & N_{1,0,k-1} & \xrightarrow{ext_u^{1,0,k-1}} & N_{0,1,k-1} \\
& & & \nearrow & & & \uparrow FL_{0,1,k-1} \\
& & & N_{k-2,0,2} & & & \\
& & \nearrow \mathcal{R}_u^{k-1} & & & & \\
& N_{k-1,0,1} & \xrightarrow{ext_u^{k-1,0,1}} & N_{k-2,1,1} & & & \\
& \uparrow FL_{k-1,1,0} & & \uparrow FL_{k-2,2,0} & & & \\
N_{k,0,0} & \xrightarrow{ext_u^{k,0,0}} & N_{k-1,1,0} & \xrightarrow{ext_u^{k-1,1,0}} & N_{k-2,2,0} & \cdots & N_{0,k,0} \\
& & & & & \nearrow & \\
& & & & & N_{0,0,k} & 
\end{array}$$

First notice that the upper triangles commute, i.e. for  $M \in D_{qc}^b(\mathcal{D}_{\mathbb{P} \times k})$  and  $u = \delta, 1, 1/t$  resp.  $M \in D_{coh}^b(\mathcal{D}_{\mathbb{P} \times k})$  and  $u = Y$  we have the following isomorphisms

$$\tilde{\mathcal{R}}_u^i(M) \simeq M \diamond \tilde{R}_u^i \simeq M \diamond (\tilde{S}_u^i \diamond L_i) \simeq (M \diamond \tilde{S}_u^i) \diamond L_i \simeq FL_{i-1,1,k-i}(M \diamond \tilde{S}_u^i) \simeq FL_{i-1,1,k-i} \circ ext_u^{i,0,k-i}(M).$$

**Lemma 2.10.** *Let  $M \in D_{qc}^b(\mathcal{D}_{\mathbb{P} \times k})$ . We have the following isomorphisms*

$$FL_{0,1,k-1} \circ ext_u^{1,0,k-1} \dots FL_{k-1,1,0} \circ ext_u^{k,0,0}(M) \simeq FL \circ ext_u^{1,k-1,0} \circ \dots \circ ext_u^{k,0,0}(M)$$

*Proof.* To prove the proposition we have to show that the following squares commute

$$\begin{array}{ccc}
N_{i,j-1,l+1} & \xrightarrow{ext_u^{i,j-1,l+1}} & N_{i-1,j,l+1} \\
\uparrow FL_{i,j,l} & & \uparrow FL_{i-1,j+1,l} \\
N_{i,j,l} & \xrightarrow{ext_u^{i,j,l}} & N_{i-1,j+1,l}
\end{array}$$

for  $i, j, l \in \mathbb{N}_0$  with  $i + j + l = k$ . It is enough to prove the commutativity of the following diagram:

$$\begin{array}{ccc}
X \times \mathbb{P} \times \hat{V} & \xrightarrow{\text{ext}_u^2} & X \times V \times \hat{V} \\
\uparrow FL_2 & & \uparrow FL_1 \\
X \times \mathbb{P} \times V & \xrightarrow{\text{ext}_u^1} & X \times V \times V
\end{array}$$

where  $X$  is some smooth algebraic variety and the functors  $FL_1, FL_2, \text{ext}_u^1, \text{ext}_u^2$  are defined below. Consider the following maps

$$\begin{aligned}
p_1 : X \times \mathbb{P} \times V \times V &\longrightarrow X \times \mathbb{P} \times V, \\
(x, [v^1], v^2, v^3) &\mapsto (x, [v^1], v^2), \\
p_2 : X \times \mathbb{P} \times V \times V &\longrightarrow X \times V \times V, \\
(x, [v^1], v^2, v^3) &\mapsto (x, v^2, v^3), \\
p : X \times \mathbb{P} \times V \times V &\longrightarrow \mathbb{P} \times V, \\
(x, [v^1], v^2, v^3) &\mapsto ([v^1], v^3).
\end{aligned}$$

The functor  $\text{ext}_u^1$  is defined by:

$$\text{ext}_u^1(M) := p_{2+}(p_1^+(M) \overset{L}{\otimes} p^+(\tilde{S}_u)).$$

Consider the following maps

$$\begin{aligned}
q_1 : X \times \mathbb{P} \times V \times \hat{V} &\longrightarrow X \times \mathbb{P} \times \hat{V}, \\
(x, [v^1], v^2, \lambda^1) &\mapsto (x, [v^1], \lambda^1), \\
q_2 : X \times \mathbb{P} \times V \times \hat{V} &\longrightarrow X \times V \times \hat{V}, \\
(x, [v^1], v^2, \lambda^1) &\mapsto (x, v^2, \lambda^1), \\
q : X \times \mathbb{P} \times V \times \hat{V} &\longrightarrow \mathbb{P} \times V, \\
(x, [v^1], v^2, \lambda^1) &\mapsto ([v^1], v^2).
\end{aligned}$$

The functor  $\text{ext}_u^2$  is defined by:

$$\text{ext}_u^2(M) = q_{2+}(q_1^+(M) \overset{L}{\otimes} q^+(\tilde{S}_u)).$$

Now define the maps

$$\begin{aligned}
\chi_1 : X \times V \times V \times \hat{V} &\longrightarrow X \times V \times V, \\
(x, v^1, v^2, \lambda^1) &\mapsto (x, v^1, v^2), \\
\chi_2 : X \times V \times V \times \hat{V} &\longrightarrow X \times V \times \hat{V}, \\
(x, v^1, v^2, \lambda^1) &\mapsto (x, v^2, \lambda^1), \\
\chi : X \times V \times V \times \hat{V} &\longrightarrow V \times \hat{V}, \\
(x, v^1, v^2, \lambda^1) &\mapsto (v^1, \lambda^1).
\end{aligned}$$

The functor  $FL_1$  is defined by:

$$FL_1(M) := \chi_{2+}(\chi_1^+(M) \overset{L}{\otimes} \chi^+ L).$$

Now define

$$\begin{aligned} \kappa_1 : X \times \mathbb{P} \times V \times \hat{V} &\longrightarrow X \times \mathbb{P} \times V, \\ (x, [v^1], v^2, \lambda^1) &\mapsto (x, [v^1], v^2), \\ \kappa_2 : X \times \mathbb{P} \times V \times \hat{V} &\longrightarrow X \times \mathbb{P} \times \hat{V}, \\ (x, [v^1], v^2, \lambda^1) &\mapsto (x, [v^1], \lambda^1), \\ \kappa : X \times \mathbb{P} \times V \times \hat{V} &\longrightarrow V \times \hat{V}, \\ (x, [v^1], v^2, \lambda^1) &\mapsto (v^2, \lambda^1). \end{aligned}$$

The functor  $FL_2$  is defined by:

$$FL_2(M) := \kappa_{2+}(\kappa_1^+(M) \overset{L}{\otimes} \kappa^+ L).$$

Consider the following diagram

$$\begin{array}{ccccc} & & X \times \mathbb{P} \times V \times V \times \hat{V} & & \\ & \swarrow \theta_1 & & \searrow \theta_2 & \\ X \times \mathbb{P} \times V \times V & & & & X \times V \times V \times \hat{V} \\ \swarrow p_1 & & \searrow p_2 & \swarrow \chi_1 & \searrow \chi_2 \\ X \times \mathbb{P} \times V & & X \times V \times V & & X \times V \times \hat{V} \end{array}$$

where the maps  $\theta_1$  and  $\theta_2$  are defined as follows

$$\begin{aligned} \theta_1 : X \times \mathbb{P} \times V \times V \times \hat{V} &\longrightarrow X \times \mathbb{P} \times V \times V, \\ (x, [v^1], v^2, v^3, \lambda^1) &\mapsto (x, [v^1], v^2, v^3), \\ \theta_2 : X \times \mathbb{P} \times V \times V \times \hat{V} &\longrightarrow X \times V \times V \times \hat{V}, \\ (x, [v^1], v^2, v^3, \lambda^1) &\mapsto (x, v^2, v^3, \lambda^1). \end{aligned}$$

Notice that the square is a cartesian diagram. We have

$$\begin{aligned}
FL_1 \circ ext_u^1(M) &= \chi_{2+} \left( \chi_1^+ p_{2+} \left( p_1^+(M) \overset{L}{\otimes} p^+ \tilde{S}_u \right) \overset{L}{\otimes} \chi^+ L \right) \\
&= \chi_{2+} \left( \theta_{2+} \theta_1^+ \left( p_1^+(M) \overset{L}{\otimes} p^+ \tilde{S}_u \right) \overset{L}{\otimes} \chi^+ L \right) \\
&= \chi_{2+} \left( \theta_{2+} \left( \theta_1^+ p_1^+(M) \overset{L}{\otimes} \theta_1^+ p^+ \tilde{S}_u \right) \overset{L}{\otimes} \chi^+ L \right) \\
&= \chi_{2+} \theta_{2+} \left( \left( \theta_1^+ p_1^+(M) \overset{L}{\otimes} \theta_1^+ p^+ \tilde{S}_u \right) \overset{L}{\otimes} \theta_2^+ \chi^+ L \right) \\
&= (\chi_2 \circ \theta_2)_+ \left( \left( (p_1 \circ \theta_1)^+(M) \overset{L}{\otimes} (p \circ \theta_1)^+ \tilde{S}_u \right) \overset{L}{\otimes} (\chi \circ \theta_2)^+ L \right).
\end{aligned}$$

Now consider the other diagram

$$\begin{array}{ccccc}
& & X \times \mathbb{P} \times V \times V \times \hat{V} & & \\
& \swarrow \rho_1 & & \searrow \rho_2 & \\
X \times \mathbb{P} \times V \times \hat{V} & & & & X \times \mathbb{P} \times V \times \hat{V} \\
\swarrow \kappa_1 & & \searrow \kappa_2 & \swarrow q_1 & \searrow q_2 \\
X \times \mathbb{P} \times V & & X \times \mathbb{P} \times \hat{V} & & X \times V \times \hat{V}
\end{array}$$

where the maps  $\rho_1$  and  $\rho_2$  are defined as follows:

$$\begin{aligned}
\rho_1 : X \times \mathbb{P} \times V \times V \times \hat{V} &\longrightarrow X \times \mathbb{P} \times V \times \hat{V}, \\
(x, [v^1], v^2, v^3, \lambda^1) &\mapsto (x, [v^1], v^2, \lambda^1), \\
\rho_2 : X \times \mathbb{P} \times V \times V \times \hat{V} &\longrightarrow X \times \mathbb{P} \times V \times \hat{V}, \\
(x, [v^1], v^2, v^3, \lambda^1) &\mapsto (x, [v^1], v^3, \lambda^1).
\end{aligned}$$

This makes the square a cartesian diagram. We have

$$\begin{aligned}
ext_u^2 \circ FL_2(M) &= q_{2+} \left( q_1^+ \kappa_{2+} \left( \kappa_1^+(M) \overset{L}{\otimes} \kappa^+ L \right) \overset{L}{\otimes} q^+ \tilde{S}_u \right) \\
&\simeq q_{2+} \left( \rho_{2+} \rho_1^+ \left( \kappa_1^+(M) \overset{L}{\otimes} \kappa^+ L \right) \overset{L}{\otimes} q^+ \tilde{S}_u \right) \\
&\simeq q_{2+} \left( \rho_{2+} \left( \rho_1^+ \kappa_1^+(M) \overset{L}{\otimes} \rho_1^+ \kappa^+ L \right) \overset{L}{\otimes} q^+ \tilde{S}_u \right) \\
&\simeq q_{2+} \rho_{2+} \left( \left( \rho_1^+ \kappa_1^+(M) \overset{L}{\otimes} \rho_1^+ \kappa^+ L \right) \overset{L}{\otimes} \rho_2^+ q^+ \tilde{S}_u \right) \\
&\simeq (q_2 \circ \rho_2)_+ \left( \left( (\kappa_1 \circ \rho_1)^+(M) \overset{L}{\otimes} (\kappa \circ \rho_1)^+ L \right) \overset{L}{\otimes} (q \circ \rho_2)^+ \tilde{S}_u \right).
\end{aligned}$$

Notice that we have  $q_2 \circ \rho_2 = \chi_2 \circ \theta_2$ ,  $\kappa_1 \circ \rho_1 = p_1 \circ \theta_1$ ,  $\kappa \circ \rho_1 = \chi \circ \theta_2$  and  $q \circ \rho_2 = p \circ \theta_1$ . This, together with the associativity of  $\overset{L}{\otimes}$  shows

$$\text{ext}_u^2 \circ FL_2 = FL_1 \circ \text{ext}_u^1.$$

But as observed above, this shows the claim.  $\square$

**Lemma 2.11.** *Let  $M \in D_{qc}^b(\mathcal{D}_{\mathbb{P} \times k})$  and  $u = \delta, 1, 1/t$  resp.  $M \in D_{coh}^b(\mathcal{D}_{\mathbb{P} \times k})$  and  $u = Y$ . We have the following isomorphisms*

$$\text{ext}_u^{1,k-1,0} \circ \dots \circ \text{ext}_u^{k,0,0}(M) \simeq \text{ext}_u^k(M).$$

where  $\text{ext}_u^k$  are the extension functors defined in Section 2.2.

*Proof.* We begin by proving the case  $u = 1$ . Set  $M_{i,j} := \mathbb{P}^{\times i} \times \tilde{V}^{\times(k-i-j)} \times V^{\times j}$  for  $i + j = k$  and define morphisms

$$\begin{aligned} \sigma_1^{ij} &:= id_{\mathbb{P}}^{\times i} \times \tilde{\pi} \times id_{\tilde{V}}^{\times k-i-j-1} \times id_V^{\times j} : M_{i,j} \longrightarrow M_{i+1,j}, \\ \sigma_2^{ij} &:= id_{\mathbb{P}}^{\times i} \times id_{\tilde{V}}^{\times k-i-j-1} \times \tilde{j} \times id_V^{\times j} : M_{i,j} \longrightarrow M_{i,j+1}, \end{aligned}$$

where  $\tilde{\pi} : \tilde{V} \rightarrow \mathbb{P}$  and  $\tilde{j} : \tilde{V} \rightarrow V$  are the maps from equation (2.1.1).

Consider the following diagram

$$\begin{array}{ccccccc} M_{0,0} & \xrightarrow{\sigma_2^{0,0}} & M_{0,1} & \cdots & M_{0,k-1} & \xrightarrow{\sigma_2^{0,k-1}} & M_{0,k} \\ \sigma_1^{0,0} \downarrow & & \sigma_1^{0,1} \downarrow & & \sigma_1^{0,k-1} \downarrow & & \\ M_{1,0} & \xrightarrow{\sigma_2^{1,0}} & M_{1,1} & \cdots & M_{1,k-1} & & \\ \vdots & & \vdots & & \vdots & & \\ M_{k-1,0} & \xrightarrow{\sigma_2^{k-1,0}} & M_{k-1,1} & & & & \\ \sigma_1^{k-1,0} \downarrow & & & & & & \\ M_{k,0} & & & & & & \end{array}$$

where the squares are cartesian diagrams. Notice that  $M_{k-i,i} = N_{k-i,i,0} = \mathbb{P}^{\times(k-i)} \times V^{\times i}$  and

$$\text{ext}_1^{k-i,i,0}(M) \simeq (\sigma_2^{k-i-1,i})_+ (\sigma_1^{k-i-1,i})_+ (M)$$

for  $i \in \{0, \dots, k-1\}$  (cf. [DE03, lemma 1]). We therefore conclude by base change that

$$\begin{aligned} \text{ext}_1^{1,k-1,0} \dots \circ \text{ext}_1^{k,0,0}(M) &\simeq (\sigma_2^{0,k-1})_+ \dots (\sigma_2^{0,0})_+ (\sigma_1^{0,0})_+ \dots (\sigma_1^{k-1,0})_+ (M) \\ &\simeq (\sigma_2^{0,k-1} \circ \dots \circ \sigma_2^{0,0})_+ (\sigma_1^{k-1,0} \circ \dots \circ \sigma_1^{0,0})_+ (M). \end{aligned}$$



Notice that  $\sigma_1^{k-1,0} \circ \dots \circ \sigma_1^{0,0} = \tilde{\pi}_k : \tilde{V}^{\times k} \rightarrow \mathbb{P}^{\times k}$  and  $\sigma_2^{0,k-1} \circ \dots \circ \sigma_2^{0,0} = \tilde{j}_k : \tilde{V}^{\times k} \rightarrow V^{\times k}$ . This shows the claim for  $u = 1$ .

The proof for  $u = Y$  and  $u = 1/t$  is virtually the same if one defines  $M_{i,j} := \mathbb{P}^{\times i} \times \dot{V}^{\times(k-i-j)} \times V^{\times j}$  with

$$\begin{aligned}\sigma_1^{ij} &:= id_{\mathbb{P}}^{\times i} \times \pi \times id_{\dot{V}}^{\times k-i-j-1} \times id_V^{\times j} : M_{i,j} \longrightarrow M_{i+1,j}, \\ \sigma_2^{ij} &:= id_{\mathbb{P}}^{\times i} \times id_{\dot{V}}^{\times k-i-j-1} \times j \times id_V^{\times j} : M_{i,j} \longrightarrow M_{i,j+1},\end{aligned}$$

where  $\pi : \dot{V} \rightarrow \mathbb{P}$  is the canonical projection and  $j : \dot{V} \rightarrow V$  the natural inclusion. In this case

$$\begin{aligned}ext_{1/t}^{k-i,i,0}(M) &\simeq (\sigma_2^{k-i-1,i})_+(\sigma_1^{k-i-1,i})^+(M), \\ ext_Y^{k-i,i,0}(M) &\simeq (\sigma_2^{k-i-1,i})_{\dagger}(\sigma_1^{k-i-1,i})^+(M)\end{aligned}$$

as well as  $\sigma_1^{k-1,0} \circ \dots \circ \sigma_1^{0,0} = \pi_k : \dot{V}^{\times k} \rightarrow \mathbb{P}^{\times k}$  and  $\sigma_2^{0,k-1} \circ \dots \circ \sigma_2^{0,0} = j_k : \dot{V}^{\times k} \rightarrow V^{\times k}$ .

It remains to prove the case  $u = \delta$ . Define  $L_{i,j} := \mathbb{P}^{\times i} \times \{0\}^{k-i-j} \times V^{\times j}$  for  $i + j \leq k$  and define morphisms

$$\begin{aligned}\alpha_1^{ij} &:= id_{\mathbb{P}}^{\times i-1} \times p \times id_{\{0\}}^{\times k-i-j} \times id_V^{\times j} : L_{i,j} \longrightarrow L_{i-1,j}, \\ \alpha_2^{ij} &:= id_{\mathbb{P}}^{\times i} \times id_{\{0\}}^{\times k-i-j-1} \times i_0 \times id_V^{\times j} : L_{i,j} \longrightarrow L_{i,j+1},\end{aligned}$$

where  $p : \mathbb{P} \rightarrow \{0\}$  is the map to the point and  $i_0 : \{0\} \rightarrow V$  is the canonical inclusion.

Consider the following diagram

$$\begin{array}{ccccccc}L_{0,0} & \xrightarrow{\alpha_2^{0,0}} & L_{0,1} & \cdots & L_{0,k-1} & \xrightarrow{\alpha_2^{0,k-1}} & L_{0,k} \\ \alpha_1^{1,0} \uparrow & & \alpha_1^{1,1} \uparrow & & \alpha_1^{1,k-1} \uparrow & & \\ L_{1,0} & \xrightarrow{\alpha_2^{1,0}} & L_{1,1} & \cdots & L_{1,k-1} & & \\ \vdots & & \vdots & & \vdots & & \\ L_{k-1,0} & \xrightarrow{\alpha_2^{k-1,0}} & L_{k-1,1} & & & & \\ \alpha_1^{k,0} \uparrow & & & & & & \\ L_{k,0} & & & & & & \end{array}$$

where the squares are commutative diagrams. For the following isomorphisms, notice that  $L_{k-i,i} = N_{k-i,i,0} = \mathbb{P}^{\times(k-i)} \times V^{\times i}$ , and recall the following maps

$$\begin{aligned}\pi_1^{k-i,i,0} &: N_{k-i,i+1,0} \longrightarrow N_{k-i,i,0} = L_{k-i,i}, \\ \pi_2^{k-i,i,0} &: N_{k-i,i+1,0} \longrightarrow N_{k-i-1,i+1,0} = L_{k-i-1,i+1}, \\ \pi^{k-i,i,0} &: N_{k-i,i+1,0} \longrightarrow \mathbb{P} \times V,\end{aligned}$$

which were defined in equation (2.3.4), (2.3.5) and (2.3.6).

and denote by  $\xi_i : L_{k-i,i} \rightarrow N_{k-i,i+1,0}$  the embedding with image  $\mathbb{P}^{\times(k-i)} \times \{0\} \times V^{\times i}$ .

$$\begin{aligned}
ext_{\delta}^{k-i,i,0}(M) &\simeq (\pi_2^{k-i,i,0})_+ \left( (\pi_1^{k-i,i,0})^+ M \otimes^L (\pi^{k-i,i,0})^+ \tilde{i}_+ \tilde{S}_{\delta} \right) \\
&\simeq (\pi_2^{k-i,i,0})_+ \left( (\pi_1^{k-i,i,0})^+ M \otimes^L \xi_{i+} \mathcal{O}_{L_{k-i,i}} \right) \\
&\simeq (\pi_2^{k-i,i,0})_+ \xi_{i+} \left( \xi_i^+ (\pi_1^{k-i,i,0})^+ M \otimes^L \mathcal{O}_{L_{k-i,i}} \right) \\
&\simeq (\pi_2^{k-i,i,0})_+ \xi_{i+} M \\
&\simeq (\alpha_2^{k-i-1,i})_+ \alpha_{1+}^{k-i,i}(M).
\end{aligned} \tag{2.3.7}$$

for  $i \in \{0, \dots, k-1\}$ , where the second isomorphism follows from the fact that the image of the exceptional divisor  $E \subset \tilde{V}$  under the embedding  $\tilde{i} : \tilde{V} \rightarrow \mathbb{P} \times V$  is equal to  $\mathbb{P} \times \{0\}$  and the fourth isomorphism follows from the fact that  $\xi_i$  is a section of  $\pi_1^{k-i,i,0}$ . We therefore conclude by the commutativity of the diagram that

$$\begin{aligned}
ext_{\delta}^{1,k-1,0} \circ \dots \circ ext_{\delta}^{k,0,0}(M) &\simeq (\alpha_2^{0,k-1})_+ \dots (\alpha_2^{0,0})_+ \alpha_{1+}^{1,0} \dots \alpha_{1+}^{k,0}(M) \\
&\simeq (\alpha_2^{0,k-1} \circ \dots \circ \alpha_2^{0,0})_+ (\alpha_1^{1,0} \circ \dots \circ \alpha_1^{k,0})_+ M \\
&\simeq (i_{0,k})_+ (p_k)_+ M \\
&\simeq (j_{0,k})_+ (\pi_k^E)^+ M \\
&= ext_{\delta}^k(M),
\end{aligned}$$

where  $p_k : \mathbb{P}^{\times k} \rightarrow \{0\}$  is the map to a point and  $i_{0,k} : \{0\} \rightarrow V^{\times k}$  is the embedding with image  $\{0\} \times \dots \times \{0\}$  and  $j_{0,k}$  resp.  $\pi_k^E$  were defined in section 2.2. □

We are now able to give the proof of Proposition 2.4.

*Proof of proposition 2.4.* The statement of the proposition follows from the following isomorphisms:

$$\begin{aligned}
\mathcal{R}'_{\tilde{u}}(M) &\simeq \tilde{\mathcal{R}}_{\tilde{u}}(\Delta_+(M)) \\
&\simeq \tilde{\mathcal{R}}_{\tilde{u}}^1 \circ \dots \circ \tilde{\mathcal{R}}_{\tilde{u}}^k(\Delta_+(M)) \\
&\simeq FL_{0,1,k-1} \circ ext_u^{1,0,k-1} \dots FL_{k-1,1,0} \circ ext_u^{k,0,0}(\Delta_+(M)) \\
&\simeq FL \circ ext_u^{1,k-1,0} \circ \dots \circ ext_u^{k,0,0}(\Delta_+(M)) \\
&\simeq FL \circ ext_u^k(\Delta_+(M)),
\end{aligned}$$

where the first isomorphism follows from Lemma 2.6, the second isomorphism from Lemma 2.8, the third isomorphism follows from Proposition 2.9, the fourth isomorphism follows from Lemma 2.10 and the last isomorphism from Lemma 2.11. □

We can now give the proof of Theorem 2.5.

*Proof of Theorem 2.5.* Using Proposition 2.4 it remains to prove

$$r^+ \mathcal{R}'_{\hat{u}}(M) \simeq \pi^+ \mathcal{R}_{\hat{u}}(M)$$

Let  $Z'' := \{[v], \lambda^1, \dots, \lambda^k \in \mathbb{P} \times S(k, n) \mid \lambda^1(v) = \dots = \lambda^k(v) = 0\}$  and  $U'' := \{[v], \lambda^1, \dots, \lambda^k \in \mathbb{P} \times S(k, n) \mid \lambda^1(v) \neq 0, \dots, \lambda^k(v) \neq 0\}$ .

We will prove the case  $r^+ \mathcal{R}'_{\delta}(M) \simeq \pi^+ \mathcal{R}_{\delta}(M)$  the other cases are similar or simpler.

Consider the following commutative diagram

$$\begin{array}{ccccc}
 & & Z' & \xrightarrow{\pi_2^{Z'}} & \hat{W} \\
 & \swarrow \pi_1^{Z'} & \uparrow r_{Z''} & & \uparrow r \\
 \mathbb{P} & \xleftarrow{\pi_1^{Z''}} & Z'' & \xrightarrow{\pi_2^{Z''}} & S(k, n) \\
 & \searrow \pi_1^Z & \downarrow \pi_{Z''} & & \downarrow \pi \\
 & & Z & \xrightarrow{\pi_2^Z} & \mathbb{G}
 \end{array}$$

where  $\pi_1^{Z''}$  resp.  $\pi_2^{Z''}$  are the canonical projections restricted to  $Z''$  and  $r_{Z''}$  resp.  $\pi_{Z''}$  were chosen such that the upper resp. lower square becomes cartesian. One has

$$\begin{aligned}
 r^+ \mathcal{R}'_{\delta}(M) &= r^+ \pi_{2+}^{Z'} (\pi_1^{Z'})^+ M \\
 &\simeq \pi_{2+}^{Z''} r_{Z''}^+ (\pi_1^{Z'})^+ M \\
 &\simeq \pi_{2+}^{Z''} (\pi_1^{Z''})^+ M \\
 &\simeq \pi_{2+}^{Z''} \pi_{Z''}^+ (\pi_1^Z)^+ M \\
 &\simeq \pi^+ \pi_{2+}^Z (\pi_1^Z)^+ M \\
 &= \pi^+ \mathcal{R}_{\delta} M
 \end{aligned}$$

□

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